Capital income taxation and economic growth under limited stock market participation

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Abstract

We examine the effect of capital income tax on the degree of stock market participation and economic growth in an economy with limited stock market participation (i.e., an economy in which only a fraction of households own stocks directly). We construct an endogenously growing overlapping generations (OLG) model, in which individuals can choose between two types of savings (i.e., physical capital with high returns but high holding costs and bank deposits with low returns but no associated costs), and banks allocate deposits between physical capital investments (such as lending to firms) and non-productive lending (such as consumption loans). Our findings show that when the bank's ratio of physical capital investments to deposits (denoted by e) is given exogenously, higher capital income taxes reduce the share of individuals who save in physical capital (i.e., reduce the degree of stock market participation), leading to a lower proportion of physical capital investment in aggregate savings and hampering economic growth. Additionally, when banks choose e endogenously, the relationship between the capital income tax rate and the share of individuals who save in physical capital can be inverted U-shaped. However, higher capital income taxes still result in a lower proportion of physical capital investment in aggregate savings, thereby hindering economic growth.

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1 Introduction

It is well known that in developed countries, only a fraction of households own stocks; that is, their participation in stock markets is limited (e.g., Mankiw and Zeldes, 1991; Haliassos and Bertaut, 1995; Campbell, 2006).[1] The reason often pointed out for limited stock market participation is differences in financial literacy (the ability to properly gather and analyze investment information) among individuals. Holding stocks requires a higher level of financial literacy than holding bank deposits does. Because acquiring financial literacy comes at a cost, it is reasonable to abandon participation in the stock market if its cost is high. Many empirical studies show that low financial literacy reduces the rate of stock market participation significantly (e.g., van Rooijet al 2011; Yoong, 2011; Thomas and Spataro, 2018).[2] This study investigates how capital income tax affects economic growth in an economy with limited stock market participation.

The relationship between capital income taxation and economic growth has been debated for a long time. In general, capital income taxation affects households' savings decisions (and hence, economic growth) by distorting the rate of return on savings, termed *the intertemporal distortionary effect of capital income tax*, hereafter. In infinitely-lived agent models, the intertemporal distortionary effect of capital income tax is widely acknowledged to hinder economic growth if the tax revenue is used for unproductive spending or transferred to households (e.g., Rebelo,1991; Jones and Manuelli, 1992; Pecorino, 1993).[3][4]

This study examines a different source of distortionary effect of capital income tax on growth from the aforementioned intertemporal distortionary effects of capital income tax, focusing on the impact of this tax on households' participation in the stock market.[5] When stock market par-

[1]Thomas and Spataro (2018) report that the average stock market participation rate of households for 9 European countries in year 2010 is 16.84%. Fujiki et al. (2012) confirm that in Japan at most 15% of households held stocks for the period from 2007 to 2010. See also Guiso et al. (2003) and Christelis et al. (2010) for other data.

[2]See also Guiso and Jappelli (2005) and Christelis et al. (2010). The former (resp. the latter) study investigates the relationship between financial awareness (resp. cognitive abilities) and stock market participation.

[3]Besides the positive analysis on the growth effect of capital income tax, a huge body of literature exists on the normative analysis of capital income taxation (i.e., optimal taxation), originally initiated by Judd (1985) and Chamley (1986). They demonstrated that a zero capital income tax is optimal in the long-run steady state.

[4]By contrast, in an overlapping generations (OLG) model, the intertemporal distortionary effect of capital income tax does not necessarily impede economic growth even when the tax revenue is used for unproductive spending. See Uhlig and Yanagawa (1996) for this point.

[5]A recent study by Jaimovich and Rebelo (2017) also assesses another source of distortionary effect of capital income tax on growth. They show that the tax reduces incentives to be entrepreneurs and causes them to stop innovating, thus distorting growth . They also show that the growth effect of a capital income tax is nonlinear when the entrepreneurial ability varies among individuals. This is because in a low-tax (resp. high-tax) economy, the ability of the marginal entrepreneur is low (resp. high), so increasing the tax rate leads to an exit of low-ability (resp.

ticipation is limited, only those with high financial literacy can participate in the stock market (i.e., only those types of individuals can have direct access to physical capital), and those with low financial literacy are forced to save in the form of bank deposits. Because the banking sector allocates deposits between physical capital investments (such as lending to firms) and non-productive lending (such as loans to consumers and holdings of government bonds), a decrease in the share of individuals who save in physical capital (in other words, an increase in the share of individuals who save in bank deposits) could increase non-productive lending in aggregate savings and hinder economic growth. In such an economy, a higher capital income tax would burden physical capital holders and decrease the share of physical capital holders, thereby hampering economic growth. This study aims to present a model in which such a hypothetical scenario of the impact of capital income tax can hold.

To achieve this objective, we develop an endogenously growing OLG model that incorporates two types of savings options for individuals: physical capital with high returns but high holding costs, and bank deposits with low returns but no associated costs. For the sake of clarity, we first consider in Section 2 the case in which the ratio of bank's physical capital investments to deposits (denoted by e in this paper) is given exogenously, and then in Section 3, the case in which e is endogenous (i.e., the bank optimally determines e). To illustrate a hypothetical scenario of the impact of capital income tax, we consider the case of Cobb-Douglas (log-linear) utility function and eliminate the usual "intertemporal distortionary effect of capital income taxes".[6]

We then show the following results: First, if e is exogenous, the hypothetical scenario described above can hold. In other words, when capital income taxes rise, the share of individuals who save in physical capital (i.e., the degree of stock market participation) falls because of the relatively large tax burden on physical capital holders. This reduces the proportion of physical capital investment in aggregate savings, thereby impeding economic growth.

Second, when e is endogenous, the above result changes slightly. In this case, higher capital income taxes have two opposite effects on the share of physical capital holders. On the one hand, it decreases the share of physical capital holders for the same reasons as when e is exogenous. On the other hand, it also increases banks' non-productive lending (i.e., the supply of consumer loans) and lowers the loan interest rate. The latter effect occurs when e is endogenous, and

high-ability) entrepreneurs.

[6]In an OLG model, the usual "intertemporal distortionary effect of capital income taxes" is known to disappear under the Cobb-Douglas (log-linear) utility function. See Uligh and Yanagawa (1996) for this point.

such decreases in the loan interest rate raise the share of physical capital holders by lowering the deposit interest rate. These opposite effects can result in an inverted U-shaped relationship between the capital income tax rate and the share of physical capital holders. Even with this non-monotonic effect on the share of physical capital holders, the effect of higher capital income taxes on economic growth is still negative. This is because even if higher capital income taxes increase the share of physical capital holders (contrary to the case where e is exogenous), the endogenous decline of e by banks still results in a lower proportion of physical capital investment in aggregate savings.

Finally, we note that our conclusion (i.e., that (higher) capital income taxes impede economic growth) is not due to the usual "intertemporal distortionary effect of capital income taxes." This is because we demonstrate this result under the Cobb-Douglas (log-linear) utility function that eliminates the intertemporal distortionary effect of capital income taxes. We show that this conclusion stems from the negative effect of capital income taxes on the proportion of physical capital investment in aggregate savings.

2 The basic model

In this section, we set up an endogenous growth model with limited stock market participation based on Maebayashi and Tanaka (2022). Our model has the following two main features. First, individuals consider their financial literacy and choose between two types of savings (i.e., physical capital with high returns but high holding costs, and bank deposits with low returns but no associated costs). Second, banks allocate deposits between physical capital investments (i.e., lending to firms) and non-productive loans (i.e., lending to consumers).

2.1 Firms

The production sector is composed of homogenous firms and the total number of firms is normalized to one. Each firm produces final goods using two types of production factors (i.e., physical capital and labor). We assume that both the final goods and factor markets are competitive, and the price of the final goods is normalized to 1 (i.e., the final goods are numeraire). The production function of each firm is given by:

$$Y_t = AK_t^{\alpha}(h_t L_t)^{1-\alpha} \qquad 0 < \alpha < 1, \tag{1}$$

where Y_t, K_t, L_t, A and h_t are the output of the final goods, physical capital input, labor input, total factor productivity, and labor-augmenting productivity, respectively. We assume that capital depreciates fullly in one period. Following Romer (1986), the labor-augmenting productivity h_t is specified as

$$h_t = K_t, \tag{2}$$

and each firm maximizes its profit by taking $h_t(\equiv K_t)$ as given. The first-order conditions for profit maximization are:

$$\alpha AK_t^{\alpha-1}(h_t L_t)^{1-\alpha} = 1 + r_t^k, \quad (1-\alpha)AK_t^{\alpha}h_t(h_t L_t)^{-\alpha} = w_t \tag{3}$$

Here, r_t^k and w_t are the (net) rate of return on physical capital and the wage rate, respectively.

2.2 Individuals

Each individual lives for two periods (young and old). The set of individuals born in period t is called " generation t ". The population of each generation is assumed to be 1.

There are two types of individuals in each generation: those who save (hereafter, savers) and those who do not (hereafter, spenders). The population of savers is $0 < \lambda < 1$. They work only in their young age, save a portion of their wages, and use the returns from saving for old-age consumption. Savers with high financial literacy save in the form of physical capital, whereas those with low financial literacy save in the form of bank deposits. In contrast, the population of spenders is $1 - \lambda$. They work in both young and old periods, borrow for consumption when young, and repay debt in their old period.[7] The reason for introducing spenders into our model is to depict a situation where a portion of aggregate savings is allocated (through banks) to non-

[7]Macroeconomic analysis that considers the households of such hand-to-mouth type has been introduced by Mankiw (2000) in a neoclassical growth model. See also the studies by Gali et al. (2004, 2007) in the literature of the New Keynesian models.

productive lending, such as consumption loans. By considering this situation, we shed light on the importance of increasing stock market participation (the share of individuals who save in physical capital) for economic growth. Capital income taxes can reduce this incentive and hamper economic growth, which constitutes the main result of this study.

2.2.1 Savers

Savers choose one of two types of savings instruments: physical capital and bank deposits, and they face the following trade-off between these two. In the former case, the gross rate of return on physical capital (denoted by $R^k_{t+1}(\equiv 1 + r^k_{t+1})$) is higher than that on bank deposits (denoted by $R^d_{t+1}(\equiv 1 + r^d_{t+1})$); however, the actual rate of return on physical capital that individuals can receive depends on the level of financial literacy.[8] (A more detailed explanation of this point is provided in the next paragraph.) In the latter case, the gross rate of return on bank deposits is lower than that on physical capital; however, no financial literacy is required to hold bank deposits.

Savers are endowed with one unit of labor when young, and they supply it inelastically. When saver i saves in the form of physical capital, he/she allocates a proportion ($\phi_{i,t}$) of his/her after-tax wage income ($(1 - \tau_L)w_t$) to the expenditure to acquire financial literacy and allocates the rest to young-age consumption (c^y_t) and savings (s_t). Therefore, the budget constraint in his/her young period is $c^y_{i,t} + s_{i,t} = (1 - \phi_{i,t})(1 - \tau_L)w_t$. As mentioned, the actual return from savings depends on the level of financial literacy. Specifically, we assume that individual i can only receive fraction $1 - \beta\sigma_i/\phi_{i,t}$ of the total return $R^k_{t+1}s_{i,t}$ (where β is a positive constant). In other words, the saver fails to receive $(\beta\sigma_i/\phi_{i,t})R^k_{t+1}s_{i,t}$.[9] The unrecovered amount $(\beta\sigma_i/\phi_{i,t})R^k_{t+1}s_{i,t}$ is smaller when $\phi_{i,t}$ is larger and σ_i is smaller. Here, σ_i is an exogenous parameter that represents saver i's learning ability, and a smaller σ_i corresponds to a higher learning ability. We assume that σ_i is uniformly distributed in the interval $[\underline{\sigma}, \overline{\sigma}]$ and satisfies $0 < \underline{\sigma} \leq \sigma_i \leq \overline{\sigma} < 1/\beta$.[10] Accordingly, the saver with the highest (resp. lowest) learning ability has $\underline{\sigma}$ (resp. $\overline{\sigma}$).

[8]This assumption is often employed when modeling the individual's investment behavior in financial literacy. See Jappelli and Padula (2013) and Lusardi et al. (2017).

[9]The resources that individuals fail to receive are assumed to be disposed of without being used for either consumption or investment.

[10]See (5) and Figure 1 regarding the reason for setting the range of σ_i to $0 < \sigma_i < 1/\beta$.

We assume that the utility function of individual i in generation t is given by

$$U_{i,t} = \left(c_{i,t}^{y}\right)^{a} \left(c_{i,t+1}^{o}\right)^{1-a}, \quad 0 < a < 1,$$

where $c_{i,t}^{y}$ and $c_{i,t+1}^{o}$ are the young- and old-age consumption, respectively. Each saver compares the utility level derived from choosing physical capital with that from choosing bank deposits and then chooses the one that gives higher utility.[11] The utility maximization problem when choosing physical capital is:

$$\max_{s_{i,t},\phi_{i,t}} U_{i,t}^{k} \text{ s.t. } c_{i,t}^{y} + s_{i,t} = (1-\phi_{i,t})(1-\tau_L)w_t, \;\; c_{i,t+1}^{o} = (1-\tau_K)(1-\beta\sigma_i/\phi_{i,t})R_{t+1}^{k}s_{i,t},$$

where $U_{i,t}^{k}$, τ_L, and τ_K are the utility level when the saver chooses physical capital, labor income tax, and capital income tax rates, respectively. Throughout this study, both τ_L and τ_K are assumed to be constant over time. Note that capital income tax is imposed on the return on savings that individual i actually receives (i.e., $(1-\beta\sigma_i/\phi_{i,t})R_{t+1}^{k}s_{i,t}$).

By arranging the first-order conditions, we have:

$$\left(\frac{\partial U_{i,t}}{\partial s_{i,t}} = 0\right) \quad s_{i,t} = (1-a)(1-\phi_{i,t})(1-\tau_L)w_t, \tag{4a}$$

$$\left(\frac{\partial U_{i,t}}{\partial \phi_{i,t}} = 0\right) \quad \sigma_i = \frac{(\phi_{i,t})^2}{\beta[(1-a)(1-\phi_{i,t})+\phi_{i,t}]}, \;\; \frac{\partial \sigma_i}{\partial \phi_{i,t}} > 0, \;\; \frac{\partial^2 \sigma_i}{\partial (\phi_{i,t})^2} > 0, \;\; 0 < \frac{\beta\sigma_i}{\phi_{i,t}} < 1. \tag{4b}$$

Note here that the level of individual savings does not depend on the capital income tax τ_K (see (4a)). The assumption of a Cobb-Douglas utility function eliminates "the intertemporal distortionary effect of capital income tax," as explained in the Introduction section. However, as we demonstrate later, the change in τ_K affects economic growth not by altering individual saving itself but by changing the allocation of aggregate savings between physical capital investment and non-productive lending.

[11]Based on essentially the same idea, Spataro and Corsini (2017) formulate the portfolio selection problem of households and investigate the factors that determine the holding of risky assets. However, their model is not a macroeconomic model.

From (4b), function $\phi_{i,t}(\sigma_i)$ satisfies the following properties:

$$\phi_{i,t}(\sigma_i) = \phi(\sigma_i), \quad \phi(0) = 0, \quad \phi(1/\beta) = 1, \ \phi'(\sigma_i) > 0, \ \phi''(\sigma_i) < 0. \tag{5}$$

Figure 1 depicts the function $\phi_{i,t}(\sigma_i) = \phi(\sigma_i)$. This figure shows that savers with higher learning abilities (i.e., lower σ_i) spend a smaller proportion of their wage income on acquiring financial literacy.

[Figure 1]

By calculating the indirect utility function in this case, we have

$$U_{i,t}^k = \tilde{a}\left[(1-\tau_K)R_{t+1}^k\right]^{1-a}\left[1-\phi(\sigma_i)\right]\left(1-\frac{\beta\sigma_i}{\phi(\sigma_i)}\right)^{1-a}(1-\tau_L)w_t, \tag{6}$$

where $\tilde{a} \equiv a^a(1-a)^{1-a}$.

On the contrary, the utility maximization problem when the saver chooses bank deposits is

$$\max_{d_{i,t}} U_{i,t}^d \quad \text{s.t.} \quad c_{i,t}^y + d_{i,t} = (1-\tau_L)w_t, \ \ c_{i,t+1}^o = R_{t+1}^d d_{i,t}.$$

Here, $d_{i,t}$ is the savings in the form of bank deposits and R_{t+1}^d is the gross rate of return on bank deposits. The savings and indirect utility in this case are

$$d_{i,t}(=d_t) = (1-a)(1-\tau_L)w_t \tag{7a}$$

$$U_{i,t}^d(=U_t^d) = \tilde{a}\left(R_{t+1}^d\right)^{1-a}(1-\tau_L)w_t \tag{7b}$$

Note that $d_{i,t}$ and $U_{i,t}^d$ do not depend on an individual's index, i.

Saver i chooses physical capital if $U_{i,t}^k > U_t^d$ and bank deposits if $U_{i,t}^k < U_t^d$. Therefore, the threshold of the learning ability (σ_t^*) at which $U_{i,t}^k = U_t^d$ holds can be calculated as follows:

$$R_{t+1}^d = (1-\tau_K)R_{t+1}^k\Phi(\sigma_t^*) \quad \text{where} \quad \Phi(\sigma_t^*) \equiv \left[1-\phi(\sigma_t^*)\right]^{\frac{1}{1-a}}\left(1-\frac{\beta\sigma_t^*}{\phi(\sigma_t^*)}\right). \tag{8}$$

Regarding the function $\Phi(\sigma_t^*)$ in (8), the following properties hold (see Appendix A):

$$\Phi'(\sigma_t^*) < 0, \quad 0 < \Phi(\overline{\sigma}) < \Phi(\sigma_t^*) < \Phi(\underline{\sigma}) < 1. \tag{9}$$

If σ_i is lower (resp. higher) than the threshold (σ_t^*), saver i chooses physical capital (resp. bank deposits). Hence, the share of savers holding physical capital is $(\sigma_t^* - \underline{\sigma})/(\overline{\sigma} - \underline{\sigma})$), whereas the share of individuals holding bank deposits is $(\overline{\sigma} - \sigma_t^*)/(\overline{\sigma} - \underline{\sigma})$.

2.2.2 Spenders (hand-to-mouth consumers)

Spenders are endowed with one unit of labor in their young period and $0 < \psi < 1$ units of labor in their old period, and supply labor inelastically in each period. We suppose that they are myopic, in the sense that they prefer a higher level of young consumption. Specifically, they borrow from banks to consume more than their wage income when young and repay the debt when old. Thus, the budget constraints in the young and old periods are as follows.

$$c_t^{y,H} = (1-\tau_L)w_t + x_t, \quad c_{t+1}^{o,H} = (1-\tau_L)\psi w_{t+1} - R_{t+1}^L x_t \tag{10}$$

Here, $c_t^{y,H}$, $c_{t+1}^{o,H}$, x_t, R_{t+1}^L are the spender's young-age consumption, spender's old-age consumption, amount of bank loans, and gross interest rate on bank loans, respectively.

In this economy, we assume that, because of friction in the financial market, the spender can pledge at most a fraction $f(\in [0,1])$ of the maximum amount that spenders can repay.[12] Here, the maximum amount that spenders can repay is the discounted present value of after-tax wage income in the old period $(1-\tau_L)\psi w_{t+1}/R_{t+1}^L$. To make debt contracts credible, debt repayments cannot exceed the pledgeable value. Therefore, the borrowing constraint becomes

$$x_t \leq \frac{f(1-\tau_L)\psi w_{t+1}}{R_{t+1}^L}.$$

They borrow the maximum available amount of bank loans because they are myopic. Thus, the borrowing amount x_t is given by

$$x_t = \frac{f(1-\tau_L)\psi w_{t+1}}{R_{t+1}^L}. \tag{11}$$

When f is lower, large frictions exist in the financial market, indicating that a lower amount of bank loan is available. Thus, f is regarded as the degree of imperfection in the financial market.

[12] See Hart and Moore (1994) and Tirole (2006) for the foundations of this setting.

2.3 Banks

The banking sector is composed of homogenous banks and the total number of banks is normalized to one. From (7a), the total amount of bank deposits in period t is given by

$$D_t \equiv \lambda \int_{\sigma_t^*}^{\bar{\sigma}} d_{i,t} \frac{1}{\bar{\sigma} - \underline{\sigma}} d\sigma_i = \lambda(1-a)(1-\tau_L) w_t \frac{\bar{\sigma} - \sigma_t^*}{\bar{\sigma} - \underline{\sigma}}. \tag{12}$$

In this section, for simplicity, we assume that each bank invests an exogenous fraction (denoted by $0 < e < 1$) of D_t in physical capital and uses the rest to lend to spenders. The analysis of the case in which each bank chooses e optimally is presented in Section 3. Then, the profit of each bank realized in period $t+1$ is expressed as

$$\pi_{t+1}^b = (1-\tau_K) R_{t+1}^k e D_t + R_{t+1}^L (1-e) D_t - (R_{t+1}^d + \eta) D_t, \tag{13}$$

where η is the operating cost per unit deposit.[13] Note that capital income tax is levied on banks because they invest a portion of their deposits in physical capital.

Because we assume that the banking sector is competitive ($\pi_{t+1}^d = 0$), we obtain:

$$R_{t+1}^d = (1-\tau_K) R_{t+1}^k e + R_{t+1}^L (1-e) - \eta \tag{14}$$

This indicates that R_{t+1}^d (the gross interest rate on bank deposits) is the weighted average of $(1-\tau_K)R_{t+1}^k$ (the after-tax gross return rate on physical capital) and R_{t+1}^L (the gross interest rate on bank loans) minus η (the bank's operating costs).

From (8) and (14), we can rewrite the condition $U_{i,t}^k = U_t^d$ as

$$(1-\tau_K) R_{t+1}^k \Phi(\sigma_t^*) = (1-\tau_K) R_{t+1}^k e + R_{t+1}^L (1-e) - \eta.$$

In the following, we assume

$$\Phi(\sigma_t^*) - e > 0. \tag{15}$$

The implication of this assumption is as follows. An increase in τ_K results in a loss of $R_{t+1}^k \Phi(\sigma_t^*)$

[13] We assume the operating cost uses the final goods.

in the physical capital holders' savings earnings (where $\Phi(\sigma_t^*)$ represents the fraction of savings that can be recovered depending on their financial literacy levels). By contrast, an increase in τ_K results in a loss of eR_{t+1}^k in the bank deposit holders' savings earnings because they invest the proportion e of their savings in physical capital through banks. Thus, the assumption (15) implies that when τ_K increases, the loss incurred by the physical capital holders is larger than that incurred by the bank deposit holders.

2.4 Government

The government uses tax revenue from both labor and capital income taxes for unproductive government spending because productive government spending for promoting economic growth is beyond the scope of this study. Denoting labor income tax revenue in period t as T_t^L, we have

$$T_t^L = \tau_L w_t L_t. \tag{16}$$

Next, capital income tax revenue in period t (denoted by T_t^K) can be expressed as follows:

$$T_t^K = \lambda \int_{\underline{\sigma}}^{\sigma_{t-1}^*} \tau_K \left(1 - \frac{\beta\sigma_i}{\phi(\sigma_i)}\right) R_t^k s_{i,t-1} \frac{1}{\overline{\sigma} - \underline{\sigma}} d\sigma_i + \tau_{K,t} R_t^k e D_{t-1} \tag{17}$$

Here, the first term on the right-hand side of (17) refers to the tax revenue from old individuals who hold physical capital in period t and the second term refers to the tax revenue from banks. Substituting (4a) and (12) into (17), we have

$$\begin{aligned} T_t^K &= \tau_K R_t^k \frac{\lambda(1-a)(1-\tau_L)w_{t-1}}{\overline{\sigma} - \underline{\sigma}} [H(\sigma_{t-1}^*) + e(\overline{\sigma} - \sigma_{t-1}^*)] \\ \text{where } H(\sigma_{t-1}^*) &\equiv \int_{\underline{\sigma}}^{\sigma_{t-1}^*} \left(1 - \frac{\beta\sigma_i}{\phi(\sigma_i)}\right) [1 - \phi(\sigma_i)] d\sigma_i, \;\; H'(\sigma_{t-1}^*) > 0. \end{aligned} \tag{18}$$

From (18), we can see that T_t^K depends on σ_{t-1}^*. This is because the population of old individuals who hold physical capital in period t depends on σ_{t-1}^*.

Thus, from (16) and (18), the government's budget constraint in period t is:

$$G_t = T_t^L + T_t^K = \tau_L w_t L_t + \tau_K R_t^k \frac{\lambda(1-a)(1-\tau_L)w_{t-1}}{\overline{\sigma} - \underline{\sigma}} [H(\sigma_{t-1}^*) + e(\overline{\sigma} - \sigma_{t-1}^*)] \tag{19}$$

2.5 Equilibrium

Because all young individuals (whose population is 1) supply one unit of labor and old individuals of the hand-to-mouth type (whose population is $1-\lambda$) supply $0<\psi<1$ units of labor, the equilibrium condition in the labor market in period t is given by

$$L_t = 1+\psi(1-\lambda). \tag{20}$$

From (1), (2), (3), and (20), the factor prices and output in equilibrium can be calculated as:

$$R^k = \alpha\hat{A}, \quad w_t = (1-\alpha)\hat{A}K_t[1+\psi(1-\lambda)]^{-1}, \quad Y_t = \hat{A}K_t, \tag{21}$$

where $\hat{A} \equiv A[1+\psi(1-\lambda)]^{1-\alpha}$, Because our model is an endogenous growth model of AK-type, R^k is constant over time and w_t is proportional to K_t.

The equilibrium condition for the final goods market (or equivalently, the equilibrium condition between savings and investment) is given by:

$$K_{t+1} = \lambda\int_{\underline{\sigma}}^{\sigma_t^*} s_{i,t}\frac{1}{\overline{\sigma}-\underline{\sigma}}d\sigma_i + eD_t. \tag{22}$$

Substituting (4a), (12) and (21) into (22), we have

$$\begin{aligned}
&\frac{K_{t+1}}{K_t}(\equiv 1+\kappa_t) = B_1(1-\tau_L)\left[I(\sigma_t^*)+e(\overline{\sigma}-\sigma_t^*)\right], \\
&B_1 \equiv \frac{\lambda(1-a)(1-\alpha)\hat{A}}{[1+\psi(1-\lambda)](\overline{\sigma}-\underline{\sigma})}, \quad I(\sigma_t^*) \equiv \int_{\underline{\sigma}}^{\sigma_t^*}[1-\phi(\sigma_i)]d\sigma_i, \quad I'(\sigma_t^*) = 1-\phi(\sigma_t^*) > 0.
\end{aligned} \tag{23}$$

Here, $1+\kappa_t$ ($\equiv K_{t+1}/K_t$) is the gross economic growth rate.

In this study, we assume the following:

$$\frac{\partial\left[I(\sigma_t^*)+e(\overline{\sigma}-\sigma_t^*)\right]}{\partial\sigma_t^*} = 1-\phi(\sigma_t^*)-e > 0. \tag{24}$$

The implication of this assumption is as follows. Suppose individuals whose learning abilities σ_i equal σ_t^*. If they choose to save in deposits, they save $1-a$ proportion of one unit of income, and the bank allocates the proportion e of it to physical capital investment, leading to $e(1-a)$ being contributed to capital accumulation. By contrast, if they save in physical capital, they save $(1-a)[1-\phi(\sigma_t^*)]$ proportion of one unit of income, all of which contributes to capital accumulation. Thus, under assumption (24), an increase in σ_t^* (i.e., an increase in the share of physical capital holders) promotes economic growth.

Because (22) is rewritten into $\frac{\lambda(1-a)(1-\tau_L)w_{t-1}}{\overline{\sigma}-\underline{\sigma}} = \frac{K_t}{I(\sigma_{t-1}^*)+e(\overline{\sigma}-\sigma_{t-1}^*)}$ by (4a) and (12), substituting it into (19) and using (21), we can derive the government budget constraint as follows.

$$G_t = \tau_L w_t L_t + \tau_K R_t^k K_t \Theta(\sigma_{t-1}^*), \quad \Theta(\sigma_{t-1}^*) \equiv \frac{H(\sigma_{t-1}^*)+e(\overline{\sigma}-\sigma_{t-1}^*)}{I(\sigma_{t-1}^*)+e(\overline{\sigma}-\sigma_{t-1}^*)}. \tag{25}$$

Here, the function $\Theta(\sigma_{t-1}^*)$ satisfies the following properties (see Appendix B for the proof):

$$\Theta'(\sigma_{t-1}^*) < 0, \quad 0 < \Theta(\overline{\sigma}) < \Theta(\sigma_{t-1}^*) < \Theta(\underline{\sigma}) = 1. \tag{26}$$

The reason $\Theta'(\sigma_{t-1}^*) < 0$ holds is as follows. An increase in σ_{t-1}^* affects capital income tax revenue positively because it increases the share of physical capital holders in period t. Conversely, an increase in σ_{t-1}^* affects capital income tax revenue negatively because such a change reduces K_{t-1} (for a given K_t) and lowers wages (and thereby savings) in period $t-1$. Because the latter negative effect exceeds the former positive effect, $\Theta'(\sigma_{t-1}^*) < 0$ holds.

From (21) and (25), the government budget constraint can be rewritten as

$$g_t(\equiv G_t/Y_t) = \tau_L(1-\alpha) + \tau_K \alpha \Theta(\sigma_{t-1}^*). \tag{27}$$

The equilibrium condition for the bank loan market is given by

$$(1-\lambda)x_t = (1-e)D_t, \tag{28}$$

where the left-hand side (LHS, hereafter) and right-hand side (RHS, hereafter) represent the demand and supply for bank loans, respectively. By substituting (11) and (12) into (28), and arranging by (21) and (23), we can derive the equilibrium interest rate on bank loans as follows:

$$R_{t+1}^L = \frac{(1-\tau_L)B_2}{1-e}\left[\frac{I(\sigma_t^*)}{\overline{\sigma}-\sigma_t^*}+e\right] \equiv R^L(\sigma_t^*), \quad B_2 \equiv \frac{f(1-\lambda)\psi\hat{A}(1-\alpha)}{1+\psi(1-\lambda)}, \quad \frac{\partial R^L(\sigma_t^*)}{\partial \sigma_t^*} > 0. \tag{29}$$

From (29), we see that R^L_{t+1} increases as threshold σ^*_t increases. This is because an increase in σ^*_t reduces the share of bank deposit holders in period $t+1$, thus lowering the supply of bank loans.

2.6 Equilibrium analysis

In this subsection, we characterize the equilibrium dynamics and examine the growth effect of an increase in capital income tax.

Definition 1. *Given the initial state* $(K_0, D_{-1}, \sigma^*_{-1}, s_{i,-1}, d_{i,-1}, x_{-1}R^L_0)$ *where* $K_0 = \lambda \int_{\underline{\sigma}}^{\sigma^*_{-1}} s_{i,-1}\frac{1}{\overline{\sigma}-\underline{\sigma}}d\sigma_i + eD_{-1}$ *by* (22)*,* $D_{-1} \equiv \lambda \int_{\sigma^*_{-1}}^{\overline{\sigma}} d_{i,-1}\frac{1}{\overline{\sigma}-\underline{\sigma}}d\sigma_i$ *by* (12)*, and* x_{-1} *satisfies* $(1-\lambda)x_{-1} = (1-e)D_{-1}$ *by* (28)*, a competitive equilibrium in the economy where* τ_L *is exogenous is a sequence of*

$$\{c^y_{i,t}, c^o_{i,t}, c^{y,H}_t, c^{o,H}_t, s_{i,t}, d_{i,t}, \phi_{i,t}, \sigma^*_t, x_t, D_t, Y_t, G_t, K_{t+1}\}^\infty_{t=0}$$

and prices $\{R^k_t, R^d_t, R^L_{t+1}, w_t\}^\infty_{t=0}$ *such that (a) taking prices, tax rates* (τ_L, τ_K)*, and the distribution of* $\sigma_i \in [\underline{\sigma}, \overline{\sigma}]$ *as given, firms and households (savers) optimize their solutions (*(3), (4a), (4b)*, and* (7a)*), consumption and borrowing by households (spenders) follows* (10) *and* (11)*; (b) zero profit condition of the banking sector is satisfied (*(14)*); (c) government's budget is balanced (*(25)*) under the given tax rates* (τ_L, τ_K)*; (d) the threshold value* σ^*_t *satisfies* (8) *; (e) markets clear with* (20), (22)*, and* (28).[14]

Substituting (14), (21), and (29) into (8), we can obtain the following equation that determines the cutoff value of σ^*_t:

$$(1-\tau_K)\alpha\hat{A}\left[\Phi(\sigma^*_t) - e\right] = (1-e)R^L(\sigma^*_t) - \eta. \tag{30}$$

(30) indicates that σ^*_t becomes constant over time (i.e., $\sigma^*_t = \sigma^*$) because σ^*_t is the only variable that depends on time t in (30). We denote the LHS and RHS of (30) as $l(\sigma^*_t)$ and $r(\sigma^*_t)$, respectively. Then, we obtain the following proposition.

[14]Note that R^L_0 in period 0 is given exogenously. In our model, the market clearing condition for bank loans in period t determines R^L_{t+1} (see (28) and (29)); therefore, there is no condition that determines R^L_0 in period 0. Even if R^L_0 is given arbitrarily, it only affects the share of income between the old deposit holders and the old spenders in period 0 but not the equilibrium dynamics.

Proposition 1. *A unique balanced growth path (BGP) exists if the following condition holds:*

$$l(\underline{\sigma})(= (1-\tau_K)\alpha\hat{A}\left[\Phi(\underline{\sigma}) - e\right]) > r(\underline{\sigma})\,(= B_2(1-\tau_L)e - \eta)\,. \tag{31}$$

The growth rate $(1+\kappa)$ *of the BGP is given by*

$$1+\kappa = B_1(1-\tau_L)\left[I(\sigma^*) + e(\overline{\sigma} - \sigma^*)\right]. \tag{32}$$

Proof: $l(\sigma_t^*)$ and $r(\sigma_t^*)$, together with (9) and (29), yield

$$l'(\sigma_t^*) = (1-\tau_K)\alpha\hat{A}\Phi'(\sigma_t^*) < 0, \quad r'(\sigma_t^*) = (1-e)\frac{\partial R^L(\sigma_t^*)}{\partial \sigma_t^*} > 0. \tag{33}$$

We can also confirm that $l(\overline{\sigma})$, that is, the value of $l(\sigma_t^*)$ evaluated at $\sigma_t^* = \overline{\sigma}$, takes a finite value because $0 < \Phi(\overline{\sigma}) < 1$ holds (see (9)), whereas $r(\overline{\sigma})$ diverges to infinity (see (29)). Thus, a unique σ^* exists under (31). Once σ^* is determined, $1+\kappa$ is determined by (23), R^d by (8), with $R^k = \alpha\hat{A}$ ((21)), and R^L by (29). ■

Under assumption (15), the following holds.

$$l(\overline{\sigma})\left(= (1-\tau_K)\alpha\hat{A}\left[\Phi(\overline{\sigma}) - e\right]\right) > 0.$$

Accordingly, we can depict (30) as shown in Figure 2.

[Figure 2]

Before proceeding to a comparative static analysis, we check the validity of the above result using a numerical example. We set the baseline numerical values of the parameters, as listed in Table 1.

[Insert Table 1 here]

The details regarding the numerical settings of the parameters are as follows: For parameter a in the utility function, we set $a = 0.69$ because the discount factor should satisfy $(1-a)/a = 0.973^{30} \approx 0.44$ (see Song et al. (2012)). According to Trabandt and Uhlig (2011), we use $(\tau_L, \tau_K) = (0.28, 0.36)$ in the US economy. Parameter α in the production function is set to

$\alpha = 0.38$ (see Trabandt and Uhlig (2011)). The scale parameter A is set to 12.5579, which yields an after-tax gross return from physical capital of 3.24 (an annual rate of approximately 4% as in Trabandt and Uhlig (2011)). We set $\psi = 0.5$, under which the old spender works until approximately 65 years old, because we regard one period as 30 years. $\lambda = 0.8$ is based on Jappelli (1990) and Jappelli et al. (1998), both of which find that the ratio of individuals who bind liquidity condition is approximately 0.2. $(\underline{\sigma}, \overline{\sigma}) = (10^{-5}, 0.03)$, $\eta = 0.8$, and $\beta = 0.05$ are selected to satisfy (15), (24), and (31). We set $(e, f) = (0.8, 0.7)$ as a benchmark case because it yields a positive long-run growth rate of 1.2437 and the ratio of government spending to GDP of $g_{BGP} = 0.3067$. Furthermore, we obtain steady-state gross interest rates of $R^d_{BGP} = 2.8828$ and $R^L_{BGP} = 5.4625$.

Under these parameter settings, we can confirm that the analytical results obtained thus far can be reproduced. Figure 3-(a) shows that condition (24) holds, and Figure 3-(b) shows that condition (15) is satisfied because of $l(\overline{\sigma}) > 0$.

[Figure 3 here]

In the following, we examine how a change in τ_K affects the economy. Differentiating σ^* in (30) with respect to τ_K and considering (15) and (33), we have

$$\frac{\partial \sigma^*}{\partial \tau_K} = -\frac{\alpha \hat{A}[\Phi(\sigma^*) - e]}{r'(\sigma^*) - l'(\sigma^*)} < 0. \tag{34}$$

From (34), we see that an increase in capital income tax τ_K lowers the threshold σ^*(i.e., decreases the share of physical capital holders). This is because an increase in τ_K lowers the after-tax rate of return on physical capital, $(1 - \tau_K)R^k_{t+1}$, which reduces (resp. raises) the share of individuals who choose physical capital (resp. bank deposits) when saving (see (8)).

Furthermore, differentiating $1 + \kappa$ (the gross rate of economic growth) in (32) with respect to τ_K and considering (24) and (34), we obtain

$$\frac{\partial(1 + \kappa)}{\partial \tau_K} = B_1(1 - \tau_L)\frac{\partial \sigma^*}{\partial \tau_K}[1 - \phi(\sigma^*) - e] < 0. \tag{35}$$

From (35), we can see that an increase in τ_K impedes economic growth. Because this study assumes a situation in which savers live for two periods, earn wages only in their youth, and have a Cobb-Douglas utility function[15], this result is not driven by the intertemporal distortionary effect of the capital income tax (i.e., changes in the relative prices between present and future consumption). The negative growth effect of the capital income tax in (35) occurs because the tax decreases (resp. increases) the share of physical capital holders (resp. bank deposit holders). In this sense, the result shown in (35) arises from the limited stock market participation.

Summarizing the discussion in this subsection, we obtain the following proposition.

Proposition 2. *An increase in capital income tax lowers the threshold σ^* (i.e., the share of physical capital holders) under assumption* (15) *and impedes economic growth under assumption* (24).

3 Extension of the model: endogenous decisions of e by banks

In the basic model in the previous section, the parameter e (i.e., the bank's ratio of physical capital investment to deposits) was assumed to be given exogenously. In this section, we assume a more realistic situation in which banks optimally choose e, implying that changes in τ_K affect banks' lending behaviors. The objective is to examine whether this extension alters the main conclusion of the previous section. Because the behaviors of firms and individuals are the same as those in the previous section, we focus on the banking sector.

3.1 Endogenous decisions of e by banks

Banks use deposits for physical capital investment (i.e., lending to firms) and consumption loans to spenders, as in the previous sections.

We introduce the following operation cost function into the bank sector and replace the constant η in the previous section with it.

$$\eta(e_t) = \frac{\eta_1}{\omega} e_t^{\omega} + \frac{\eta_2}{\omega}(1 - e_t)^{\omega} + \eta_3. \quad (\omega > 1, \ \ \eta_1 > 0, \ \ \eta_2 > 0, \ \ \eta_3 > 0) \tag{36}$$

$$(\ \ \eta'(e_t) = \eta_1 e_t^{\omega-1} - \eta_2(1 - e_t)^{\omega-1}, \quad \eta'(\tilde{e}_t) = 0, \ \ \text{where} \ \ \tilde{e}_t \in (0, 1) \ \ )$$

Figure 4-(a) depicts (36) and implies that as more management resources are invested in lending

[15]In this sense, our model in this subsection corresponds to Section 3 in Uhlig and Yanagawa (1996) in which the utility function is log-linear and there is no intergenerational income transfer through changes in tax rates .

to firms (resp. hand-to-mouth consumers); namely, e_t approaches 1 (resp. 0), the marginal cost of lending to firms (resp, hand-to-mouth consumers) increases.

[Figure 4 here]

Thus, the bank's profit maximization problem is given by:

$$\max_{e_t} \pi^b_{t+1} = (1-\tau_K)R^k_{t+1}e_tD_t + R^L_{t+1}(1-e_t)D_t - \left[R^d_{t+1} + \eta(e_t)\right] D_t \tag{37}$$

The first-order condition of this problem is

$$\Lambda_{t+1} = \eta'(e_t), \quad \Lambda_{t+1} \equiv (1-\tau_K)R^k_{t+1} - R^L_{t+1} \tag{38}$$

where Λ_{t+1} is the gap between returns on physical capital and bank loans. From (36) and (38), we have

$$e_t = e(\Lambda_{t+1}), \quad e'(\Lambda_{t+1}) = \frac{1}{\eta''(e_t)} > 0, \tag{39}$$

where $e(\Lambda_{t+1}) > \tilde{e}_t$ (resp. $0 < e(\Lambda_{t+1}) < \tilde{e}_t$) for $0 < \Lambda_{t+1} < \eta_1$ (resp. $-\eta_2 < \Lambda_{t+1} < 0$) as represented in Figure 4-(b). (39) implies that banks raise e_t as the return gap Λ_{t+1} increases.

From (37) and (39), the bank's profit is expressed as

$$\pi^b_{t+1} = \left[\Lambda_{t+1}e(\Lambda_{t+1}) + R^L_{t+1} - R^d_{t+1} - \eta\left(e(\Lambda_{t+1})\right)\right] D_t$$

As $\pi^b_{t+1} = 0$ holds in the long run under perfect competition, the deposit interest rate R^d_{t+1} is given by

$$R^d_{t+1} = \Lambda_{t+1}e(\Lambda_{t+1}) + R^L_{t+1} - \eta\left(e(\Lambda_{t+1})\right). \tag{40}$$

Finally, the behavior of the government is the same as that in the previous section, but e in (17) (i.e., capital income tax revenue T^k_t) changes to $e(\Lambda_t)$.[16]

[16]Note that e in (17) is not $e(\Lambda_{t+1})$ but $e(\Lambda_t)$, because it is the ratio of physical capital investment to deposits D_{t-1} in period $t-1$.

3.2 Equilibrium analysis

In this subsection, we derive the equilibrium dynamics and examine the growth effect of an increase in τ_K.

The equilibrium factor prices are given by (21), as before, inducing the return gap Λ_{t+1} into

$$\Lambda_{t+1} = (1-\tau_K)\alpha\hat{A} - R^L_{t+1}, \quad \frac{\partial \Lambda_{t+1}}{\partial \tau_K} = -\alpha\hat{A} < 0, \quad \frac{\partial \Lambda_{t+1}}{\partial R^L_{t+1}} = -1 (< 0). \tag{41}$$

Regarding the equilibrium condition between saving and investment, e in (22) changes to $e(\Lambda_{t+1})$.[17] Thus, the economic growth rate (23) can be rewritten as:

$$\frac{K_{t+1}}{K_t} (\equiv 1+\kappa_t) = B_1(1-\tau_L)[I(\sigma_t^*) + e(\Lambda_{t+1})(\overline{\sigma} - \sigma_t^*)]. \tag{42}$$

The government's budget constraint is given by (19) (or (27)), as before, but e in $\Theta(\sigma^*_{t-1})$ changes to $e(\Lambda_t)$. Furthermore, e in (28) (the equilibrium condition for the bank loan market) changes to $e(\Lambda_{t+1})$: Thus, (29) can be rewritten as:

$$R^L_{t+1} = B_2(1-\tau_L)[1-e(\Lambda_{t+1})]^{-1}\left[\frac{I(\sigma_t^*)}{\overline{\sigma} - \sigma_t^*} + e(\Lambda_{t+1})\right]. \tag{43}$$

Appendix C shows that under a certain condition, $0 < e(\sigma_t^*, \tau_K) < 1$ (or, equivalently, $\underline{\sigma} < \sigma_t^* < \overline{\sigma_t^*} < \overline{\sigma}$) exists uniquely such that (43) is satisfied, which ensures the uniqueness of $R^L(\sigma_t^*, \tau_K)$ satisfying (43). Appendix C also establishes the relationship $e(\sigma_t^*, \tau_K) = e(\Lambda(\sigma_t^*, \tau_K))$. Appendix D illustrates that $R^L(\sigma_t^*, \tau_K)$ satisfying (43) has the following properties.

$$\frac{\partial R^L(\sigma_t^*, \tau_K)}{\partial \sigma_t^*} > 0, \quad \frac{R^L(\sigma_t^*, \tau_K)}{\partial \tau_K} < 0. \tag{44}$$

The main difference from (29) in Section 2.5 (where e is exogenous) is that here R^L_{t+1} depends on τ_K. When e is endogenous (i.e., $e(\Lambda(\sigma_t^*, \tau_K))$), an increase in τ_K lowers the return gap Λ_{t+1} (see (41)), which, in turn, reduces R^L_{t+1} by lowering e.

[17] Note that e in (22) is not $e(\Lambda_t)$ but $e(\Lambda_{t+1})$, because it is the ratio of physical capital investment to deposits D_t in period t.

Accordingly, $\Lambda(\sigma_t^*, \tau_K)(= (1-\tau_K)\alpha\hat{A} - R^L(\sigma_t^*, \tau_K))$ and $e(\sigma_t^*, \tau_K)(= e(\Lambda(\sigma_t^*, \tau_K)))$ satisfy

$$\frac{\partial\Lambda(\sigma_t^*, \tau_K)}{\partial\sigma_t^*} = -\frac{\partial R^L(\sigma_t^*, \tau_K)}{\partial\sigma_t^*} < 0, \quad \frac{\partial\Lambda(\sigma_t^*, \tau_K)}{\partial\tau_K} = -\alpha\hat{A} - \frac{\partial R^L(\sigma_t^*, \tau_K)}{\partial\tau_K} : ?, \tag{45}$$

$$\frac{\partial e(\sigma_t^*, \tau_K)}{\partial\sigma_t^*} = e'(\Lambda)\frac{\partial\Lambda(\sigma_t^*, \tau_K)}{\partial\sigma_t^*} < 0, \quad \frac{\partial e(\sigma_t^*, \tau_K)}{\partial\tau_K} = e'(\Lambda)\frac{\partial\Lambda(\sigma_t^*, \tau_K)}{\partial\tau_K} : ?. \tag{46}$$

Definition 2. *Given the initial state* $(K_0, D_{-1}, \sigma_{-1}^*, s_{i,-1}, d_{i,-1}, x_{-1}, e_{-1}, R_0^L)$ *where* $K_0 = \lambda\int_{\underline{\sigma}}^{\sigma_{-1}^*} s_{i,-1}\frac{1}{\overline{\sigma}-\underline{\sigma}}d\sigma_i + e_{-1}D_{-1}$ *by* (22), $D_{-1} \equiv \lambda\int_{\sigma_{-1}^*}^{\overline{\sigma}} d_{i,-1}\frac{1}{\overline{\sigma}-\underline{\sigma}}d\sigma_i$ *by* (12), *and* x_{-1} *satisfies* $(1-\lambda)x_{-1} = (1-e_{-1})D_{-1}$ *by* (28)*, a competitive equilibrium in the economy where* τ_L *is exogenous is a sequence of*

$$\{c_{i,t}^y, c_{i,t}^o, c_t^{y,H}, c_t^{o,H}, s_{i,t}, d_{i,t}, \phi_{i,t}, \sigma_t^*, e_t, x_t, D_t, Y_t, G_t, K_{t+1}\}_{t=0}^{\infty}$$

and prices $\{R_t^k, R_t^d, R_{t+1}^L, w_t\}_{t=0}^{\infty}$ *such that (a) taking prices, tax rates* (τ_L, τ_K)*, and the distribution of* $\sigma_i \in [\underline{\sigma}, \overline{\sigma}]$ *as given, firms, households (savers) and banks optimize their solutions (*(3), (4a), (4b), (7a)*, and* (38)*), consumption and borrowing by households (spenders) follows* (10) *and* (11)*; (b) the zero profit condition of the banking sector is satisfied (*(40)*); (c) government's budget is balanced (*(25)*) under the given tax rates* (τ_L, τ_K)*; (d) the threshold value* σ_t^* *satisfies* (8)*; and (e) markets clear with* (20)*,* (22)*, and* (28)*.*

From (8), (40), and $R_{t+1}^k = \alpha\hat{A}$ in (21), we can derive the equation that determines σ_t^* as:

$$(1-\tau_K)\alpha\hat{A}\Phi(\sigma_t^*) = \Lambda(\sigma_t^*, \tau_K)e(\sigma_t^*, \tau_K) + R^L(\sigma_t^*, \tau_K) - \eta(e(\sigma_t^*, \tau_K)) \tag{47}$$

We denote the LHS (resp. RHS) of (47) as $\tilde{l}(\sigma_t^*)$ (resp. $\tilde{r}(\sigma_t^*)$). Then, the following proposition holds:

Proposition 3. *A unique BGP exists under the following conditions.*

$$\tilde{r}(\underline{\sigma}) < \tilde{l}(\underline{\sigma}), \qquad \tilde{r}\left(\overline{\sigma_t^*}\right) > \tilde{l}\left(\overline{\sigma_t^*}\right) \tag{48}$$

The growth rate $(1+\kappa)$ *of the BGP is given by*

$$1+\kappa = B_1(1-\tau_L)[I(\sigma^*) + e(\sigma^*, \tau_K)(\overline{\sigma} - \sigma^*)].$$

Proof: From (47) with (38), (44), (45), and (46), we have

$$\tilde{r}'(\sigma_t^*) = [1 - e(\sigma_t^*, \tau_K)] \frac{\partial R^L(\sigma_t^*, \tau_K)}{\partial \sigma_t^*} > 0. \tag{49}$$

From (9), we can obtain

$$\tilde{l}'(\sigma_t^*) = (1 - \tau_K)\alpha \hat{A} \Phi'(\sigma_t^*) < 0, \qquad 0 < \tilde{l}(\overline{\sigma}) < \tilde{l}(\underline{\sigma}) < (1 - \tau_K)\alpha \hat{A}. \tag{50}$$

Hence, (47) can be drawn as shown in Figure 5 under condition (48), indicating that the BGP exists uniquely. ■

[Figure 5 here]

Before proceeding to a comparative static analysis, we check the validity of the above result using a numerical example. We select the values of the newly appearing parameters as follows:

$$(\eta_1, \eta_2, \eta_3, \omega) = (1.0, 5.0, 0.1, 2.0), \tag{51}$$

whereas the other parameter values remain unchanged. We select a relatively high value of η_2 that satisfies $\eta_2 > \eta_1$ (under the baseline $\eta_1 = 1$) to obtain a large value of $e(\sigma^*, \tau_K)$ (i.e., banks lend more funds to firms than households (spenders)).[18] In this benchmark case, (47) can be depicted as in Figure 6-(d), which shows that (48) is satisfied and that a unique BGP exists. In this BGP, we obtain $e^* = 0.7177$, $\sigma^* = 0.0208$, $1 + \kappa = 1.2128$, $R^d_{BGP} = 2.8795$, $R^L_{BGP} = 3.9335$, and $g_{BGP} = 0.3065$. Furthermore, $\overline{\sigma_t^*}$ defined by (C.2), is 0.0287 (Figure 6-(c)).

[Figure 6 here]

We now examine the growth effect of an increase in capital income tax. Differentiating σ^* in (47) with respect to τ_K, we obtain:

$$\frac{\partial \sigma^*}{\partial \tau_K} = -\frac{\alpha \hat{A}[\Phi(\sigma^*) - e] + (1 - e)\frac{\partial R^L}{\partial \tau_K}}{\tilde{r}'(\sigma^*) - \tilde{l}'(\sigma^*)} : (?) \tag{52}$$

When e is endogenous, an increase in τ_K affects σ^* through two distinct effects. First, an increase in τ_K reduces the share of physical capital holders because it imposes a relatively large tax burden

[18]See Figure 4-(b) to comfirm that high value of η_2 is associated with large values of $e(\sigma^*, \tau_K)$ and $\tilde{e}_t$.

on them under assumption (15). The first term in the numerator of (52) shows this effect. This effect is common to the case where e is exogenous (see (34)). Second, when τ_K rises, banks increase the supply of non-productive loans (i.e., loans to spenders), which lowers the loan interest rate R^L. This, in turn, lowers the deposit interest rate R^d and reduces the share of bank deposit holders (in other words, increases the share of physical capital holders). The second term in the numerator of (52) shows this effect and is specific to the case where e is endogenous.

These two opposing effects on σ^* make the sign of (52) ambiguous in general. Figure 7-(a) shows the numerically calculated effects of $\partial\sigma^*/\partial\tau_K$ under the parameter settings in Table 1 and (51), leading to the following result: Under relatively low (resp. high) values of τ_K, the sign of $\partial\sigma^*/\partial\tau_K$ is positive (resp. negative). In other words, the relationship between the capital income tax rate and the share of physical capital holders is inverted U-shaped.

Next, we examine the effect of an increase in τ_K on the growth $(1+\kappa)$. From Proposition 3, along with (24), (46), and (52), we obtain

$$\frac{\partial(1+\kappa)}{\partial\tau_K} = B_1(1-\tau_L)\Bigg[\underbrace{\frac{\partial\sigma^*}{\partial\tau_K}[1-\phi(\sigma^*)-e]}_{(\#1)} + \Bigg(\underbrace{\frac{\partial e}{\partial\tau_K}}_{(\#2)} + \underbrace{\frac{\partial e}{\partial\sigma^*}\frac{\partial\sigma^*}{\partial\tau_K}}_{(\#3)}\Bigg)(\overline{\sigma}-\sigma^*)\Bigg] : (?). \quad (53)$$

When e is endogenous, an increase in τ_K affects $1+\kappa$ through two distinct effects. First, an increase in τ_K affects economic growth by changing the share of physical capital holders. The term $(\#1)$ in (53) illustrates this effect. The sign of $\partial\sigma^*/\partial\tau_K$ is ambiguous from (52), which renders the sign of this effect ambiguous even under assumption (24). Second, an increase in τ_K affects economic growth through an endogenous change in e. Banks allocate deposits between physical capital investments and non-productive loans. An increase in τ_K not only reduces the after-tax rate of return on physical capital $(1-\tau_K)\alpha\hat{A}$, but also has direct and indirect impacts on the loan interest rate R^L. Banks set e considering changes in these return rates, which, in turn, affects economic growth. The terms $(\#2)$ and $(\#3)$ in (53) illustrate these effects. The signs of these effects are also generally ambiguous.

However, Figure 7-(b) shows that under the parameter settings in Table 1 and (51), an increase in τ_K lowers $1+\kappa$, which is common to the case when e is exogenous (see (35)). Numerical studies also demonstrate that other ambiguous properties of variables in (45) and (46) are $\partial\Lambda(\sigma^*,\tau_K)/\tau_K < 0$ (Figure 7-(c)) and $\partial e(\sigma^*,\tau_K)/\tau_K < 0$ (Figure 7-(d)).

These numerical results indicate the following. On the one hand, when $\partial\sigma^*/\partial\tau_K < 0$ in Figure 7-(a) (i.e., when a higher τ_K reduces the share of physical capital holders), the negative growth effect $(\#1)$ in (53) becomes relatively stronger. On the other hand, when $\partial\sigma^*/\partial\tau_K > 0$ in Figure 7-(a) (i.e., when a higher τ_K raises the share of physical capital holders), the negative growth effect of $(\#2)$ and $(\#3)$ in (53) becomes relatively stronger. Consequently, irrespective of whether $\partial\sigma^*/\partial\tau_K$ is positive or negative, a higher τ_K hinders economic growth.

[Figure 7 here]

Summarizing the discussion in this subsection, we obtain the following proposition:

Proposition 4. *Suppose the bank optimally chooses e (i.e., the ratio of physical capital investments to bank deposits). The effects of an increase in τ_K (capital income tax) on both the threshold σ^* and the economic growth rate $1+\kappa$ are generally ambiguous. Under the parameter settings listed in Table 1 and* (51)*, the relationship between the capital income tax rate and the share of physical capital holders becomes inverted U-shaped. Despite this, higher capital income taxes reduce economic growth.*

4 Conclusion

We developed an endogenously growing OLG model in which (1) individuals can choose between two types of savings (i.e., physical capital with high returns but high holding costs and bank deposits with low returns but no associated costs) and (2) banks allocate deposits between physical capital investments (i.e., lending to firms) and non-productive loans (i.e., lending to spenders). Our investigation focused on the effects of capital income tax on the degree of stock market participation and economic growth.

If the bank's ratio of physical capital investments to deposits (denoted by e) is exogenous, higher capital income taxes can reduce the share of physical capital holders (i.e., reduce the degree of stock market participation). Such a change lowers the proportion of physical capital investment in aggregate savings, which can impede economic growth.

By contrast, if banks determine e endogenously, higher capital income taxes have both negative and positive effects on the share of physical capital holders. The negative effect is common to the case where e is exogenous. However, the positive effect arises because higher capital income taxes increase the supply of non-productive consumption loans by banks and then lowers the loan interest rate, which, in turn, lowers the interest rate on deposits. Consequently, the relationship between the capital income tax rate and the share of physical capital holders becomes inverted U-shaped. Even with this non-monotonic effect on the share of physical capital holders, the growth effect of the capital income tax is still negative. This is because even if higher capital income taxes increase the share of physical capital holders (contrary to the case where e is exogenous), the endogenous reduction of e by banks results in a lower proportion of physical capital investment in aggregate savings.

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Appendix

A Proof of (9)

By differentiating $\Phi(\sigma_t^*)$ in (8) with respect to σ_t^*, we have

$$\Phi'(\sigma_t^*) = -(1-\phi)^{\frac{\alpha}{1-\alpha}} \left[\phi' \frac{1}{1-\alpha} \left(1 - \frac{\beta\sigma_t^*}{\phi}\right) + (1-\phi)\beta \frac{\phi - \sigma_t^*\phi'}{\phi^2} \right]. \tag{A.1}$$

From (4b) and (5), the sign of the first term in the square brackets is positive. $\phi - \sigma_t^*\phi'$ in the second term in square brackets is also positive because function ϕ is strictly concave (see (5)). Hence, $\Phi'(\sigma_t^*) < 0$ holds true. Clearly, $0 < \Phi(\overline{\sigma}) < \Phi(\sigma_t^*) < \Phi(\underline{\sigma}) < 1$ holds, because both $0 < 1 - \phi(\sigma_t^*) < 1$ and $0 < 1 - \beta\sigma_t^*/\phi(\sigma_t^*) < 1$ are satisfied (see (4b) and (5)).

B Proof of (26)

Differentiating $\Theta(\sigma_{t-1}^*)$ in (25) with respect to σ_{t-1}^*, we obtain

$$\begin{aligned}\Theta'(\sigma_{t-1}^*) &= \frac{(H'-e)[I + e(\overline{\sigma} - \sigma_{t-1}^*)] - [H + e(\overline{\sigma} - \sigma_{t-1}^*)](I'-e)}{[I + e(\overline{\sigma} - \sigma_{t-1}^*)]^2}. \\ &= \frac{(H' \cdot I - I' \cdot H) + (H' - I')e(\overline{\sigma} - \sigma_{t-1}^*) + e(H - I)}{[I + e(\overline{\sigma} - \sigma_{t-1}^*)]^2}\end{aligned} \tag{B.1}$$

Here, we calculate $H - I$, $H' - I'$, and $H' \cdot I - I' \cdot H$ as follows:

$$\begin{aligned}H - I &= \int_{\underline{\sigma}}^{\sigma_{t-1}^*} \left(1 - \frac{\beta\sigma_i}{\phi(\sigma_i)}\right)[1 - \phi(\sigma_i)]d\sigma_i - \int_{\underline{\sigma}}^{\sigma_{t-1}^*} [1 - \phi(\sigma_i)]d\sigma_i \\ &= -\int_{\underline{\sigma}}^{\sigma_{t-1}^*} \frac{\beta\sigma_i}{\phi(\sigma_i)}[1 - \phi(\sigma_i)]d\sigma_i < 0 \end{aligned} \tag{B.2}$$

$$\begin{aligned}H' - I' &= \left(1 - \frac{\beta\sigma_{t-1}^*}{\phi(\sigma_{t-1}^*)}\right)[1 - \phi(\sigma_{t-1}^*)] - [1 - \phi(\sigma_{t-1}^*)] \\ &= -\frac{\beta\sigma_{t-1}^*}{\phi(\sigma_{t-1}^*)}[1 - \phi(\sigma_{t-1}^*)] < 0 \end{aligned} \tag{B.3}$$

$$\begin{aligned}H' \cdot I - I' \cdot H &= \left(1 - \frac{\beta\sigma_{t-1}^*}{\phi(\sigma_{t-1}^*)}\right)[1 - \phi(\sigma_{t-1}^*)] \int_{\underline{\sigma}}^{\sigma_{t-1}^*} [1 - \phi(\sigma_i)]d\sigma_i \\ &\quad - [1 - \phi(\sigma_{t-1}^*)] \int_{\underline{\sigma}}^{\sigma_{t-1}^*} \left(1 - \frac{\beta\sigma_i}{\phi(\sigma_i)}\right)[1 - \phi(\sigma_i)]d\sigma_i \\ &= [1 - \phi(\sigma_{t-1}^*)] \int_{\underline{\sigma}}^{\sigma_{t-1}^*} \left[\left(1 - \frac{\beta\sigma_{t-1}^*}{\phi(\sigma_{t-1}^*)}\right) - \left(1 - \frac{\beta\sigma_i}{\phi(\sigma_i)}\right)\right][1 - \phi(\sigma_i)]d\sigma_i \end{aligned} \tag{B.4}$$

Since the function $\phi(\sigma^*_{t-1})$ is strictly concave by (5), the following relationship holds within the range of $\sigma_i \in [\underline{\sigma}, \sigma^*_{t-1}]$ (see Figure 8):

$$\frac{\phi(\sigma_i)}{\sigma_i} > \frac{\phi(\sigma^*_{t-1})}{\sigma^*_{t-1}} > \beta \iff 0 < 1 - \frac{\beta\sigma^*_{t-1}}{\phi(\sigma^*_{t-1})} < 1 - \frac{\beta\sigma_i}{\phi(\sigma_i)} < 1 \tag{B.5}$$

This together with (B.4) indicates $H' \cdot I - I' \cdot H < 0$. Thus, $H - I < 0$, $H' - I' < 0$, and $H' \cdot I - I' \cdot H < 0$ imply $\Theta'(\sigma^*_{t-1}) < 0$.

Furthermore, from the definitions of functions $H(\sigma^*_t)$ and $I(\sigma^*_t)$ given in (18) and (23), we can easily confirm that $H(\underline{\sigma}) = I(\underline{\sigma}) = 0$. Thus, the following holds:

$$\Theta(\underline{\sigma}) = \frac{e(\overline{\sigma} - \underline{\sigma})}{e(\overline{\sigma} - \underline{\sigma})} = 1 \tag{B.6}$$

Next, noting that $0 < 1 - \beta\sigma_i/\phi(\sigma_i) < 1$ is satisfied for any $\sigma_i \in [\underline{\sigma}, \sigma^*_{t-1}]$ (see (B.5)), we obtain

$$0 < \Theta(\overline{\sigma}) = \frac{\int_{\underline{\sigma}}^{\overline{\sigma}} \left(1 - \frac{\beta\sigma_i}{\phi(\sigma_i)}\right)[1 - \phi(\sigma_i)]d\sigma_i}{\int_{\underline{\sigma}}^{\overline{\sigma}}[1 - \phi(\sigma_i)]d\sigma_i} < 1. \tag{B.7}$$

Thus, $0 < \Theta(\overline{\sigma}) < \Theta(\sigma^*_{t-1}) < \Theta(\underline{\sigma}) = 1$ holds true.

[Figure 8 here]

C Uniqueness of $e(\sigma^*_t, \tau_K)$ and $R^L(\sigma^*_t, \tau_K)$

From (36), (38), (39), (41), and (43), we have

$$B_2(1 - \tau_L)[1 - e_t]^{-1}\left[\frac{I(\sigma^*_t)}{\overline{\sigma} - \sigma^*_t} + e_t\right] (= R^L_{t+1}) = (1 - \tau_K)\alpha\hat{A} - \eta_1 e_t^{\omega-1} + \eta_2(1 - e_t)^{\omega-1}. \tag{C.1}$$

(C.1) is represented in (the upper right-hand side of) Figure 9. A unique value of $e_t (\equiv e(\sigma^*_t, \tau_K)) \in (0, 1)$ that satisfies (C.1) exists under the following condition (i.e., the LHS of (C.1) is smaller than the RHS for $e_t = 0$):

$$B_2(1 - \tau_L)\frac{I(\sigma^*_t)}{\overline{\sigma} - \sigma^*_t} < (1 - \tau_K)\alpha\hat{A} + \eta_2 \quad \Leftrightarrow \quad \sigma^*_t < \overline{\sigma^*_t}(< \overline{\sigma}), \tag{C.2}$$

where $B_2(1 - \tau_L)\frac{I(\overline{\sigma^*_t})}{\overline{\sigma} - \overline{\sigma^*_t}} = (1 - \tau_K)\alpha\hat{A} + \eta_2$.

[Figure 9]

Figure 9 also shows the uniqueness of $R^L_{t+1}(\equiv\ R^L(\sigma^*_t, \tau_K))$ by (C.1) (and (43)) and $\Lambda_{t+1}(\equiv \Lambda(\sigma^*_t, \tau_K) = (1-\tau_K)\alpha\hat{A} - R^L(\sigma^*_t, \tau_K))$ by (41). From (39), $e(\sigma^*_t, \tau_K) = e(\Lambda(\sigma^*_t, \tau_K))$ holds between $e(\sigma^*_t, \tau_K)$ and $\Lambda(\sigma^*_t, \tau_K)$.

D Derivation of (44)

By totally differentiating (41) and (43), we obtain

$$d\Lambda_{t+1} = -\alpha\hat{A}d\tau_K - dR^L_{t+1} \tag{D.1}$$

$$dR^L_{t+1} = \frac{B_2(1-\tau_L)}{1-e(\Lambda_{t+1})}\frac{\partial}{\partial\sigma^*_t}\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}\right]d\sigma^*_t + \frac{B_2(1-\tau_L)}{[1-e(\Lambda_{t+1})]^2}e'(\Lambda_{t+1})\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}+1\right]d\Lambda_{t+1}. \tag{D.2}$$

From (D.1), (D.2), and $\frac{\partial}{\partial\sigma^*_t}\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}\right] > 0$ using (23), we obtain

$$\frac{\partial R^L_{t+1}}{\partial\sigma^*_t} = \frac{\frac{B_2(1-\tau_L)}{1-e(\Lambda_{t+1})}\frac{\partial}{\partial\sigma^*_t}\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}\right]d\sigma^*_t}{1+\frac{B_2(1-\tau_L)}{[1-e(\Lambda_{t+1})]^2}e'(\Lambda_{t+1})\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}+1\right]} > 0, \tag{D.3}$$

$$\frac{\partial R^L_{t+1}}{\partial\tau_K} = -\frac{\alpha\hat{A}\frac{B_2(1-\tau_L)}{[1-e(\Lambda_{t+1})]^2}e'(\Lambda_{t+1})\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}+1\right]}{1+\frac{B_2(1-\tau_L)}{[1-e(\Lambda_{t+1})]^2}e'(\Lambda_{t+1})\left[\frac{I(\sigma^*_t)}{\overline{\sigma}-\sigma^*_t}+1\right]} < 0. \tag{D.4}$$

Therefore, (44) is obtained.

Parameter	a	α	A	ψ	β	λ	e
Value	0.69	0.38	12.5579	0.5	0.05	0.8	0.8
Parameter	f	η	τ_L	τ_K	$\overline{\sigma}$	$\underline{\sigma}$	
Value	0.7	0.8	0.28	0.36	0.03	10^{-5}	

Table 1: The benchmark numerical settings of parameters

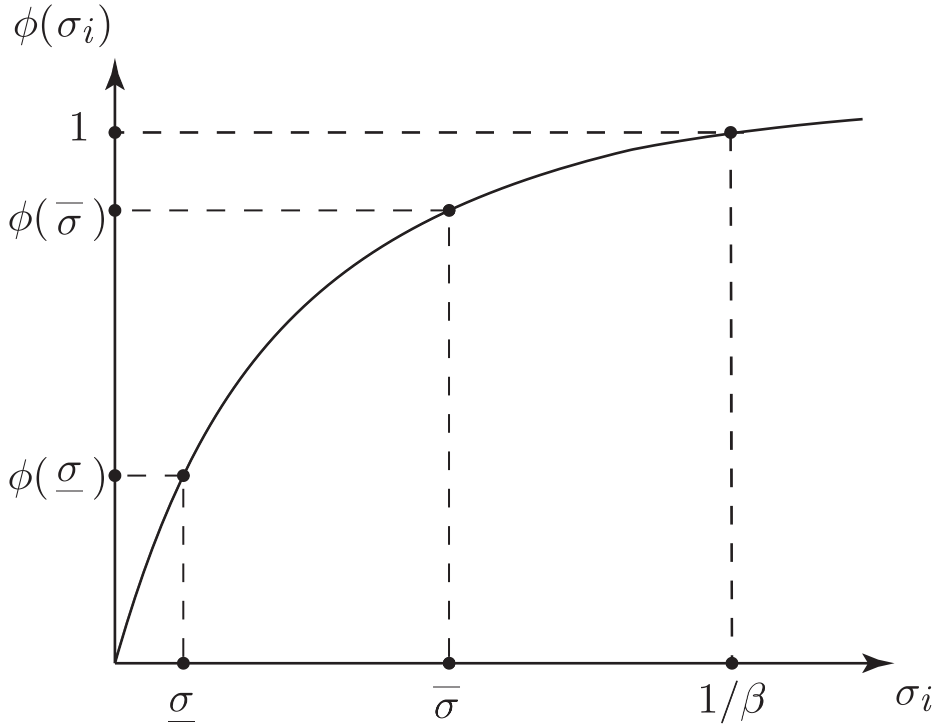

Figure 1: The function $\phi(\sigma_i)$

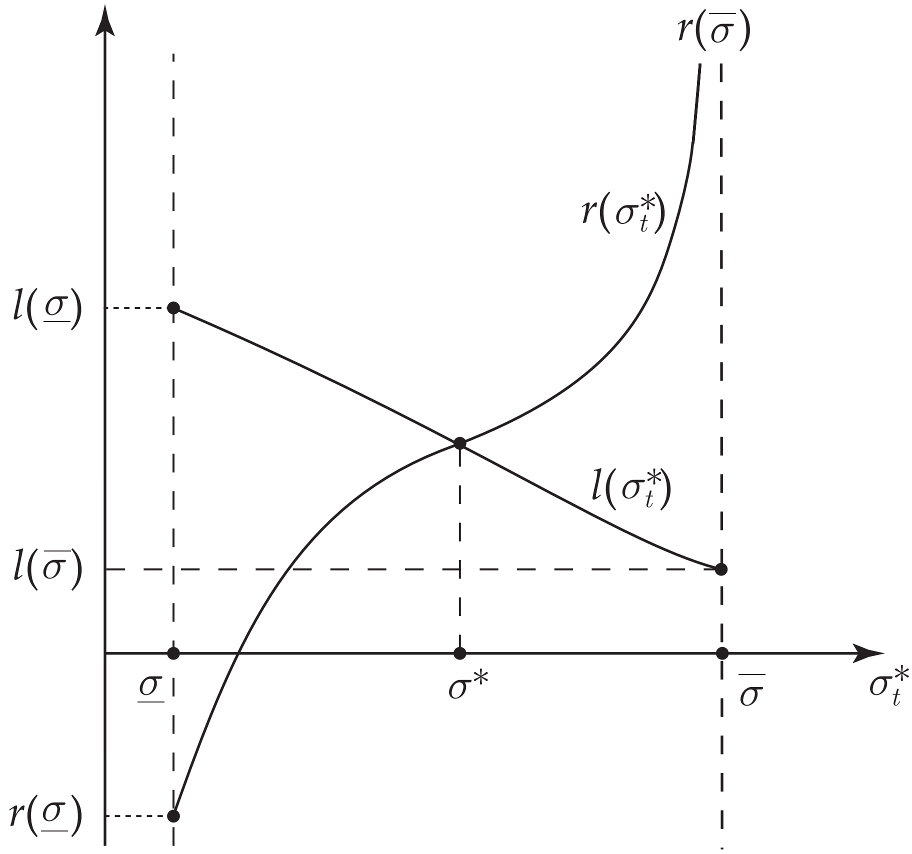

Figure 2: Uniqueness of σ^*

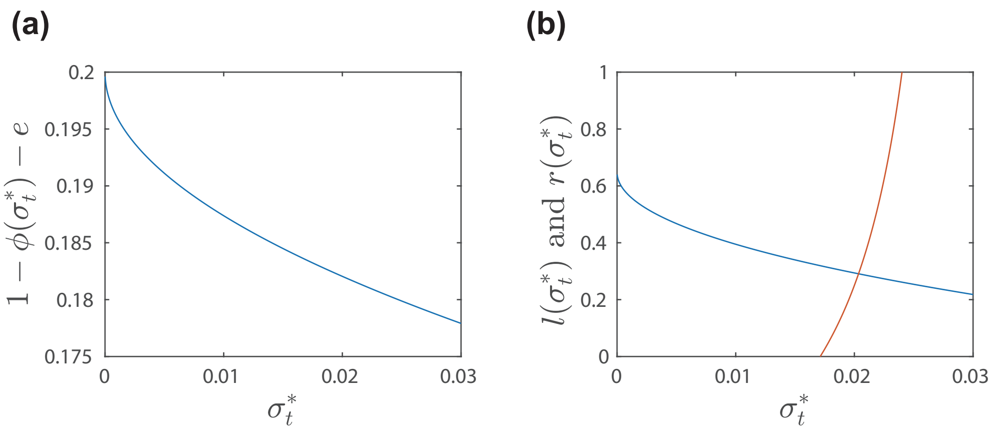

Figure 3: Numerical demonstrations of (a) Eq. (24) and (b) Eq. (30)

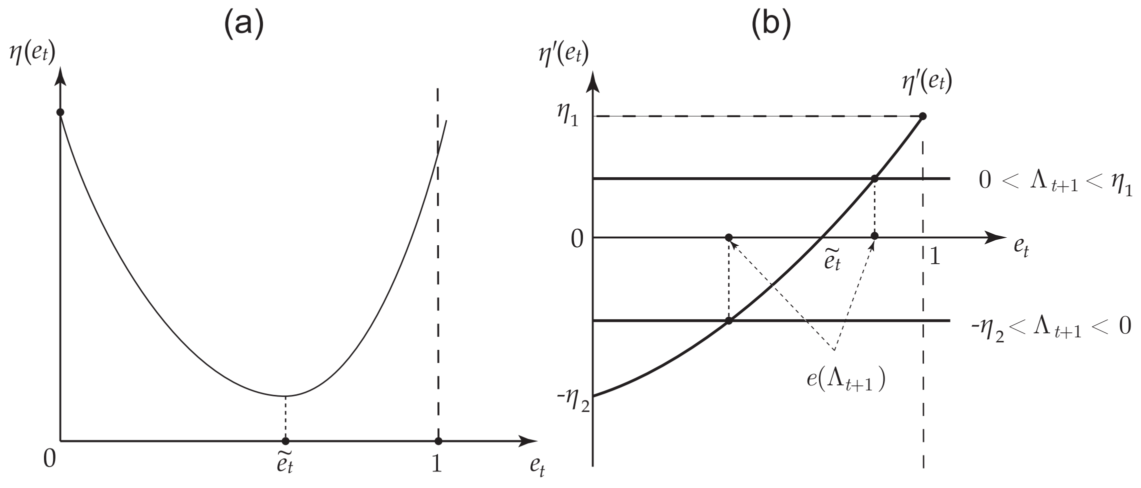

Figure 4: The cost function $\eta(e_t)$

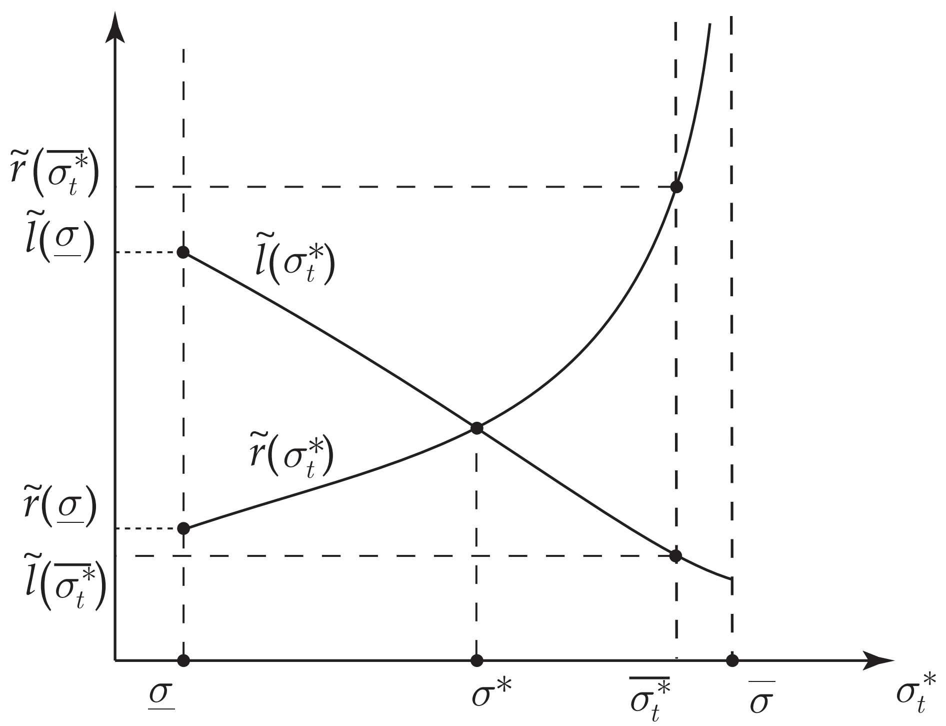

Figure 5: Uniqueness of σ^*

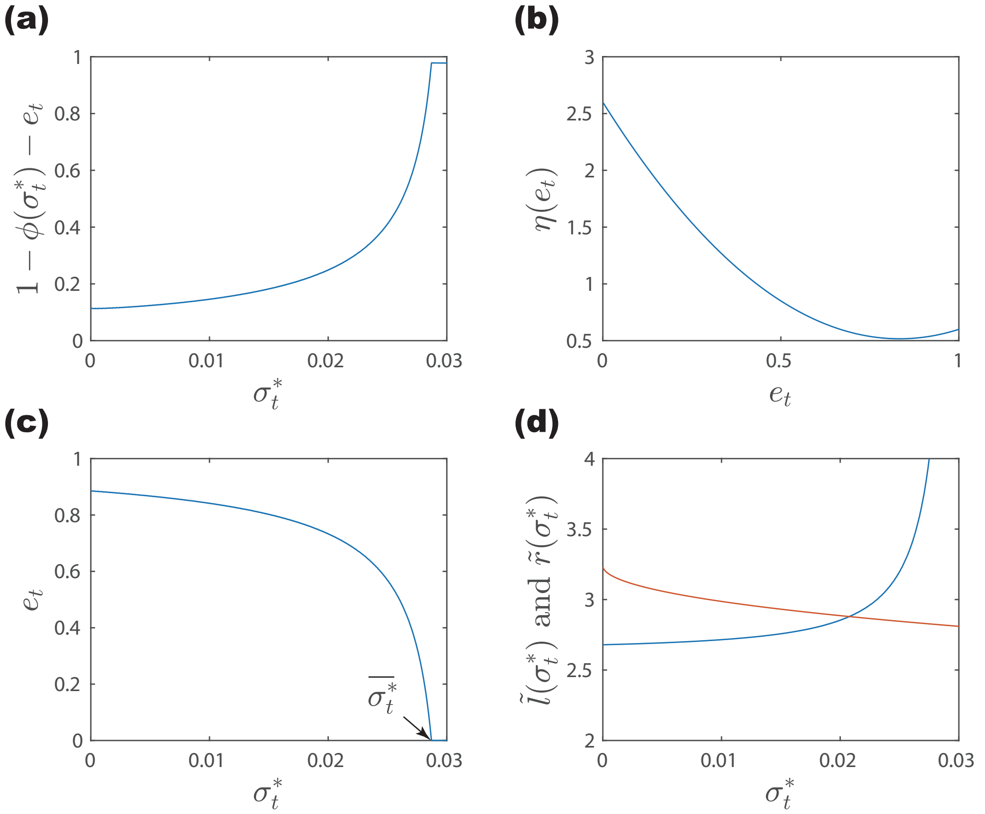

Figure 6: Numerical demonstrations of (24), $\eta(e_t)$, and the relationship between σ_t^* and e_t, $\tilde{r}(\sigma_t^*)$, and $\tilde{l}(\sigma_t^*)$

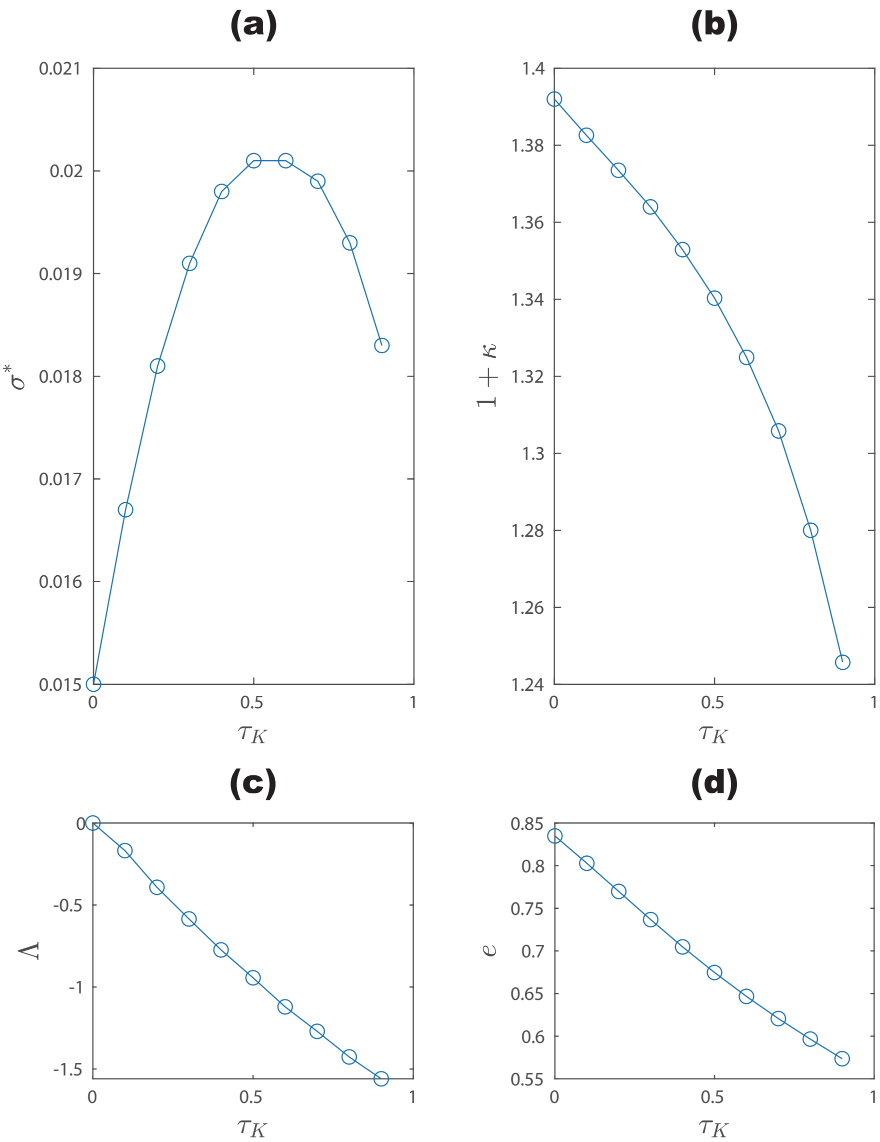

Figure 7: The relationship between τ_K and σ^*, $1+\kappa$, $\Lambda(\sigma^*, \tau_K)$, and $e(\sigma^*, \tau_K)$

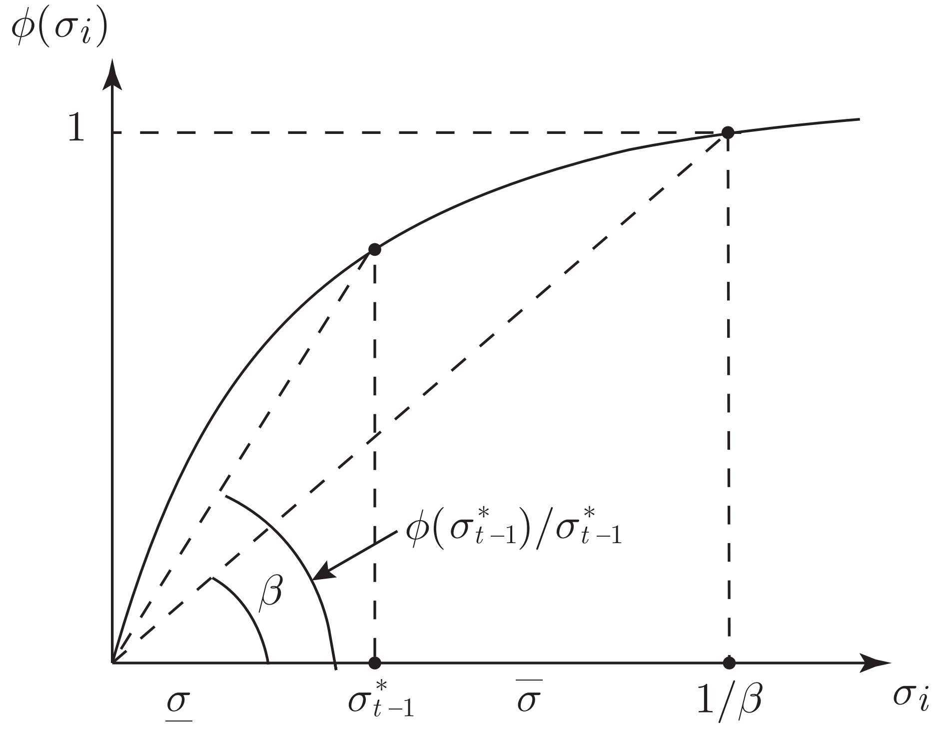

Figure 8: Representation of (B.5)

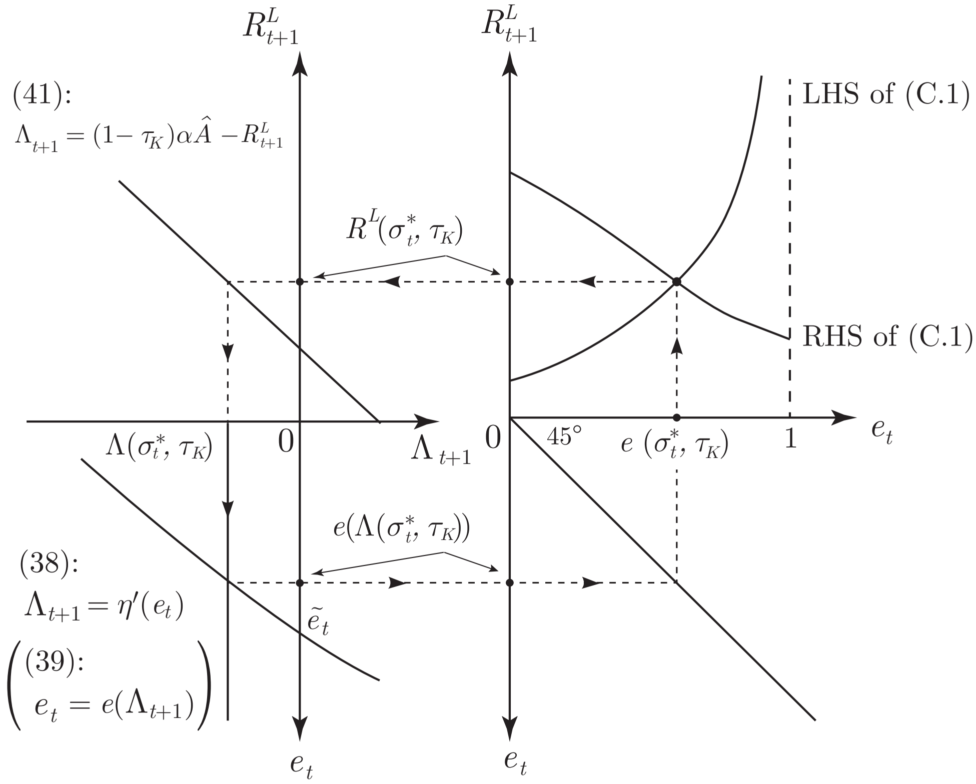

Figure 9: Uniqueness of $e(\sigma^*_t, \tau_K)$, $R^L(\sigma^*_t, \tau_K)$, and $\Lambda(\sigma^*_t, \tau_K)$